GRAPH FREE PRODUCT OF NONCOMMUTATIVE PROBABILITY SPACES

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ABSTRACT. In this paper, we will introduce a new operator-algebraic probability structure, so-called the Graph Free Product Spaces. Let G be a simplicial finite graph with its probability-space-vertices $\{(A_v, \varphi_v) : v \in V(G)\}$. Define the graph free product $A = \underset{v \in V(G)}{*^G} A_v$, as a vector space $\mathbb{C} \oplus (\underset{w \in \mathbb{F}^+(G)}{\oplus} A_w)$, where $\mathbb{F}^+(G)$ is the free semigroupoid of G, consisting of all vertices as units and all admissible finite paths, with $A_w = A_w$, if w is a vertex in G, and with $A_w = A_{v_1} * A_{v_2} * ... * A_{v_k}$, whenever $w = [v_1, ..., v_k]$ is a finite path. Also, define the canonical subalgebra D^G of A by $D^G = \mathbb{C} \oplus (\underset{w \in \mathbb{F}^+(G)}{\oplus} \mathbb{C})$. Then definitely D^G is a subalgebra of A. The algebraic pair (A, E) is a noncommutative probability space with amalgamation over D^G in the sense of Voiculescu, where $E = \underset{w \in \mathbb{F}^+(G)}{\oplus} \varphi_w$ is the conditional expectation from A onto D^G. In fact, this structure is a direct producted noncommutative probability space introduced in [9]. This amalgamated noncommutative probability space (A, E) is called the graph free product space of $\{(A_v, \varphi_v) : v \in V(G)\}$. We will consider the noncommutative probability on it. We can characterize the graph-freeness pictorially for the given graph. i.e., the subalgebras A_{w_1} and A_{w_2} are graph-free in (A, E) if and only if w_1 and w_2 are disjoint on the graph G. By using this graph-freeness, we establish the graph R-transform calculus.

In this paper, we will define and observe Graph Free Probability Spaces which are the graph free product of probability-space-vertices. Graph free product spaces are the direct sum of free products of noncommutative probability spaces, where the direct sum is highly depending on the given graphs. Roughly speaking, a graph free product space is a certain graph-depending direct sum of free product spaces. Free Probability has been developed by Voiculescu, Speicher and various mathematicians from 1980's. There are two approaches to study it. One of them is the original Voiculescu's pure analytic approach (See [4]) and the other one is the Speicher and Nica's combinatorial approach (See [1], [11] and [12]). Let $B \subset A$ be unital algebras with $1_A = 1_B$. Suppose that there exists a conditional expectation $E : A \rightarrow B$ satisfying the bimodule map property and

(i) $E(b) = b$, for all $b \in B$
(ii) $E(bab') = bE(a)b'$, for all $b, b' \in B$ and $a \in A$.

Then the algebraic pair (A, E) is called a noncommutative probability space with amalgamation over B (or an amalgamated noncommutative probability space over B. See [11]). All elements in (A, E) are said to be B-valued random variables. When $B = \mathbb{C}$ and E is a linear functional, then we call this structure a (scalar-valued) noncommutative probability space (in short, a probability space) and the elements of A are said to be (free) random variables. Let $a \in (A, E)$ be a B-valued random variable. Then it contains the following (equivalent) free probabilistic data,

$$E(b_1 a ... b_n a)$$

and

$$k_n(b_1 a, ..., b_n a) \overset{def}{=} \sum_{\pi \in NC(n)} E_\pi(b_1 a, ..., b_n a) \mu(\pi, 1_n),$$

which are called the B-valued n-th moment of a and the B-valued n-th cumulant of a, respectively, for all $n \in \mathbb{N}$ and $b_1, ..., b_n \in B$ are arbitrary, where $E_\pi (...)$ is the partition-dependent B-valued moment of a and $NC(n)$ is the collection of all noncrossing partitions over $\{1, ..., n\}$ and μ is the Möbius functional in the incidence algebra I_2, as the convolution inverse of the zeta functional ζ,

$$\zeta(\pi_1, \pi_2) \overset{def}{=} \begin{cases} 1 & \text{if } \pi_1 \leq \pi_2 \\ 0 & \text{otherwise,} \end{cases}$$

for all $\pi_1, \pi_2 \in NC(n)$, and $n \in \mathbb{N}$ (See [11]). When $b_1 = ... = b_n = 1_B$, for $n \in \mathbb{N}$, we say that $E(a^n)$ and $k_n(a, ..., a)$ are trivial n-th B-valued moment and cumulant of a, respectively. Recall that the collection $NC(n)$ is a lattice with the total ordering $\leq$ (See [1], [2], [10] and [11]). Again, if $B = \mathbb{C}$, then we have the same (scalar-valued) moments and cumulants of each random variable $a \in (A, E)$. But, in this case, since $B = \mathbb{C}$ commutes with A, we have that

$$E(b_1 a ... b_n a) = (b_1 ... b_n) E(a^n)$$

and

$$k_n(b_1 a, ..., b_n a) = (b_1 ... b_n) k_n(a, ..., a),$$

for all $n \in \mathbb{N}$ and for all $b_1, ..., b_n \in B = \mathbb{C}$. These mean that we only need to consider the trivial moments and cumulants of a, with respect to $E : A \to B = \mathbb{C}$. So, the (scalar-valued) moments and cumulants of a are defined by the trivial $B = \mathbb{C}$-valued moments and cumulants of it. More generally, if an arbitrary unital algebra B commutes with A, where A is over B, then we can verify that the trivial B-valued moments and cumulants of a B-valued random variable a contains the full free probabilistic data of a. So, in this case, without loss of generality, we define the B-valued moments and cumulants of a by the trivial B-valued moments and cumulants of a.

Let A_1 and A_2 be subalgebras of A. We say that they are free over B if all mixed cumulants of A_1 and A_2 vanish. Let S_1 and S_2 be subsets of A. We say that subsets S_1 and S_2 are free over B if the subalgebras $A_1 = A \lg \{S_1, B\}$ and $A_2 = A \lg \{S_2, B\}$ are free over B. In particular, the B-valued random variables a_1 and a_2 are free

over B if the subsets $\{a_1\}$ and $\{a_2\}$ are free over B in (A, E). Equivalently, given two B-valued random variables a_1 and a_2 are free over B in (A, E) if all mixed B-valued cumulants of a_1 and a_2 vanish. (Recall that, when A is a $*$-algebra, they are free over B if all mixed B-valued cumulants of $P(a_1, a_1^*)$ and $Q(a_2, a_2^*)$ vanish, for all P, Q in $\mathbb{C}\,[z_1,\, z_2]$.)

Let G be a finite simplicial (undirected) graph with its vertex set $V(G)$ and its edge set $E(G)$. A graph is simplicial if it has neither multiple edges between two vertices nor loop-edges. For example, the circulant graph C_N with N-vertices is a simplicial graph. Let e be an edge connecting vertices v_1 and v_2. Then we denote e by $[v_1,\, v_2]$ or $[v_2,\, v_1]$, and we assume that $[v_1,\, v_2] = [v_2,\, v_1]$. Suppose w is a finite path in the graph connecting edges $[v_1,\, v_2]$, $[v_2,\, v_3]$, ..., $[v_{k-1},\, v_k]$. Then, for convenience, we denote w by $[v_1,\, v_2,\, ...,\, v_k]$. Of course, we assume that

$$[v_1, v_2, ..., v_k] = [v_k, ..., v_2, v_1],$$

as a same finite path. In this case, the length $|w|$ of the finite path w is defined to be $k - 1$, the cardinality of the admissible edges. Define the set $FP(G)$ by the set of all such finite paths and define the free semigroupoid $\mathbb{F}^+(G)$ of the given graph G by

$$\mathbb{F}^+(G) = V(G) \cup FP(G),$$

as a set.

Let G be a finite simplicial graph with its probability-space-vertices

$$\{(A_v, \varphi_v) : v \in V(G)\},$$

where (A_v, φ_v) are noncommutative probability space with their linear functionals $\varphi_v : A_v \to \mathbb{C}$, indexed by the vertices v in $V(G)$. Define the graph free (or G-free) product A^G of A_v's by

$$A^G \overset{denote}{=} \underset{v \in V(G)}{*^G} A_v \overset{def}{=} \mathbb{C} \oplus \left(\underset{w \in \mathbb{F}^+(G)}{\oplus} A_w \right),$$

as an algebra, where $A_w = A_w$, if $w \in V(G)$, and

$$A_w = A_{[v_1, ..., v_k]} = A_{v_1} * ... * A_{v_k},$$

for all $w = [v_1, ..., v_k] \in FP(G)$, where " $*$ " is the usual (scalar-valued) free product. Similarly, define the free product of linear functional φ_w on A_w by

$$\varphi_w = \varphi_w \text{ if } w \in V(G)$$

and

$$\varphi_w = \varphi_{[v_1, ..., v_k]} = \varphi_{v_1} * ... * \varphi_{v_k},$$

if $w = [v_1, ..., v_k] \in FP(G)$. Then, for each $w \in \mathbb{F}^+(G)$, we have the free product space (A_w, φ_w), as a noncommutative probability space with its linear functional φ_w. Notice that if w_1 and w_2 are finite paths in $FP(G)$ and assume that $w_3 = w_1 w_2$ is also an admissible finite path (i.e., $w_3 \in FP(G)$). Then we have that

$$(0.1) \qquad \left(A_{w_3}, \varphi_{w_3}\right) = \left(A_{w_1}, \varphi_{w_1}\right) * \left(A_{w_2}, \varphi_{w_2}\right).$$

On the other hands, if w_1 and w_2 are finite paths in $FP(G)$ and if they are not admissible (i.e., $w_1 w_2 \notin FP(G)$ or $w_2 w_1 \notin FP(G)$). Then the free product $A_{w_1} * A_{w_2}$ of A_{w_1} and A_{w_2} does not exist as a direct summand of the graph free product $A^G = \underset{v \in V(G)}{*^G} A_v$, however, there exists a direct summand $A_{w_1} \oplus A_{w_2}$ of A^G.

Define the following canonical subalgebra D^G of A^G by

$$D^G \overset{def}{=} \mathbb{C} \oplus \left(\underset{w \in \mathbb{F}^+(G)}{\oplus} \mathbb{C}_w \right), \;\; \text{with} \;\; \mathbb{C}_w = \mathbb{C}, \forall w \in \mathbb{F}^+(G).$$

Then we can define the conditional expectation E from A^G onto D^G by $E = \underset{w \in \mathbb{F}^+(G)}{\oplus} \varphi_w$. The algebraic pair (A^G, E) is called the graph free product of probability-space vertices $\{(A_v, \varphi_v) : v \in V(G)\}$ or the G-free probability space. And all elements of this G-free product space (A^G, E) are called G-free random variables.

The main purpose of studying the G-free probability spaces is to extend the classical free probabilistic data to a certain free-product-like structure depending on a pure combinatorial object (a graph). Also, the G-free product, itself, is the extension of the usual free product. For example, if the given graph H is complete, in the sense that there always exists an edge $[v, v']$, for all pair (v, v') of vertices, then the H-free product is nothing but the classical free product.

In Chapter 1, we will review the direct producted noncommutative probability spaces introduced in [9]. In [9], we only observed the finite direct product of noncommutative probability spaces, but in Chapter 1, we will consider the infinite direct product of noncommutative probability spaces. After considering the direct producted noncommutative probability spaces, we can realize that our graph free product (A^G, E) of probability-space-vertices $\{(A_v, \varphi_v)\}_{v \in V(G)}$ is a certain countable direct producted noncommutative probability space, having its direct summands as free products of probability spaces, depending on an admissible finite path.

In Chapter 2, we consider the graph-freeness characterization. And we can characterize the graph-freeness of two subalgebras of A^G, in terms of the disjointness on the graph G. i.e., the subalgebras A_{w_1} and A_{w_2} of A^G are graph-free if and only if w_1 and w_2 are disjoint on the graph G. Also, we will define the graph moments and graph cumulants of a G-free random variable in (A^G, E). And then, we will

consider the G-freeness on A^G. Under this G-freeness, we will do the graph R-transform calculus over D^G. In Chapter 3, we will study certain G-free random variables under the W^*-setting. Finally, in Chapter 4, we will consider certain subalgebras of the graph free probability space.

1. Direct Producted Noncommutative Probability Spaces

Throughout this chapter, let's fix a sufficiently big number $N \in \mathbb{N}$ (possibly, $N \to \infty$) and the collection $\mathcal{F}$ of (scalar-valued) noncommutative probability spaces,

$$\mathcal{F} = \{(A_i, \varphi_i) : i = 1, ..., N\}.$$

Then, for the given unital algebras A_1, ..., A_N in $\mathcal{F}$, we can define the direct producted unital algebra

$$A = \oplus_{j=1}^{N} A_j = \{\oplus_{j=1}^{N} a_j : a_j \in A_j, \, j = 1, ..., N\},$$

where $\oplus$ means the algebra direct sum with its componentwise vector addition and the vector multiplication defined again componentwisely by

$$\left(\oplus_{j=1}^{N} a_j\right) \cdot \left(\oplus_{j=1}^{N} a_j'\right) = \oplus_{j=1}^{N} a_j a_j',$$

for all $\oplus_{j=1}^{N} a_j$, $\oplus_{j=1}^{N} a_j' \in A$. This direct product A of A_1, ..., A_N is again a unital algebra with its unity $1_A = \oplus_{j=1}^{N} 1$.

Define the subalgebra D_N of the direct producted algebra $A = \oplus_{j=1}^{N} A_j$ by

$$D_N \overset{def}{=} \oplus_{j=1}^{N} \mathbb{C}_j, \text{ with } \mathbb{C}_j = \mathbb{C}, \, \forall j = 1, ..., N.$$

Then D_N is a commutative subalgebra of $M_N(\mathbb{C})$ and it is isomorphic to the matricial algebra Δ_N generated by all $N \times N$ diagonal matrices in the matricial algebra $M_N(\mathbb{C})$. We will call this subalgebra D_N of A, the N-th diagonal algebra of A.

Definition 1.1. *Define the direct producted noncommutative probability space A of unital algebras A_1, ..., A_N, by the noncommutative probability space (A, E) with amalgamation over the N-th diagonal algebra D_N, where $A = \oplus_{j=1}^{N} A_j$ is the direct product of A_1, ..., A_N and $E : A \to D_N$ is the conditional expectation from A onto D_N defined by*

$$E\left(\oplus_{j=1}^{N} a_j\right) = \oplus_{j=1}^{N} \varphi_j(a_j),$$

for all $\oplus_{j=1}^{N} a_j \in A$. Some times, we will denote E by $\oplus_{j=1}^{N} \varphi_j$.

It is easy to see that the $\mathbb{C}$-linear map E is indeed a conditional expectation;

(i) $E\left(\oplus_{j=1}^{N}\alpha_j\right) = \oplus_{j=1}^{N}\varphi_j(\alpha_j) = \oplus_{j=1}^{N}\alpha_j,$

   for all $\oplus_{j=1}^{N}\alpha_j \in D_N.$

(ii) $E\left(\left(\oplus_{j=1}^{N}\alpha_j\right)\left(\oplus_{j=1}^{N}a_j\right)\left(\oplus_{j=1}^{N}\alpha_j'\right)\right)$

   $= E\left(\oplus_{j=1}^{N}\alpha_j a_j \alpha_j'\right) = \oplus_{j=1}^{N}\varphi_j(\alpha_j a_j \alpha_j')$

   $= \oplus_{j=1}^{N}\left(\alpha_j \cdot \varphi_j(a_j) \cdot \alpha_j'\right)$

   $= \left(\oplus_{j=1}^{N}\alpha_j\right) \cdot \left(\oplus_{j=1}^{N}\varphi_j(a_j)\right) \cdot \left(\oplus_{j=1}^{N}\alpha_j'\right)$

   $= \left(\oplus_{j=1}^{N}\alpha_j\right)\left(E(\oplus_{j=1}^{N}a_j)\right)\left(\oplus_{j=1}^{N}\alpha_j'\right),$

   for all $\oplus_{j=1}^{N}\alpha_j, \oplus_{j=1}^{N}\alpha_j' \in D_N$ and $\oplus_{j=1}^{N}a_j \in A.$

By (i) and (ii), the map E is a conditional expectation from $A = \oplus_{j=1}^{N} A_j$ onto D_N. Thus the algebraic pair (A, E) is a noncommutative probability space with amalgamation over the N-th diagonal algebra D_N.

Definition 1.2. *Let (A_j, φ_j) be $*$-probability spaces, for $j = 1, ..., N$, where A_j's are unital $*$-algebras and φ_j's are linear functional satisfying that $\varphi_j(a_j^*) = \overline{\varphi_j(a_j)}$, for all $a_j \in A_j$, for $j = 1, ..., N$. Then the direct producted $*$-probability space (A, E) is defined by the algebraic pair of direct product $A = \oplus_{j=1}^{N} A_j$ of $A_1, ..., A_N$, as a $*$-algebra, with $(\oplus_{j=1}^{N}a_j)^* = \oplus_{j=1}^{N}a_j^*$, for $\oplus_{j=1}^{N}a_j \in A$, and the conditional expectation satisfying the above condition (i), (ii) and the following condition (iii);*

$$E\left((\oplus_{j=1}^{N}a_j)^*\right) = E\left(\oplus_{j=1}^{N}a_j\right)^* \quad in \ D_N,$$

for all $\oplus_{j=1}^{N}a_j \in A$. Similarly, we can define the direct producted C^*-probability (or W^*-probability) spaces, if A is a C^*-direct sum (resp. W^*-direct sum) of C^*-algebras (resp. von Neumann algebras), also denoted by $\oplus_{j=1}^{N} A_j$, and the conditional expectation E satisfy (i), (ii) and (iii), with the continuity under the given C^*-topology (resp. W^*-topology) on A. (Notice that the C^* or W^*-topology of A is the product topology of those of $A_1, ..., A_N$ $(N \to \infty)$. And the continuity comes from that of $\varphi_1, ..., \varphi_N$, under the product topology $(N \to \infty)$.)

In the rest of this section, we will let $A_1, ..., A_N$ be just unital algebras without the involution and topology. However, if we put involution or topology on $A_1, ..., A_N$, we would have the same or similar results.

Now, we will consider the D_N-freeness on the direct producted noncommutative probability space (A, E). Notice that the N-th diagonal algebra D_N satisfies that

$$(1.1) \qquad dx = xd, \text{ for all } d \in D_N \text{ and } x \in A,$$

as a subalgebra of our direct product $A = \oplus_{j=1}^{N} A_j$. By (1.1) and the commutativity of D_N, we only need to consider the trivial D_N-valued moments and cumulants of D_N-valued random variables, for studying the free probabilistic data of them. i.e., we have that, for any $a \in A$,

$$(1.2) \qquad E(d_1 a...d_n a) = (d_1...d_n) \, E(a^n)$$

and

$$(1.3) \qquad k_n (d_1 a, ..., d_n a) = (d_1...d_n) \, k_n (a, ..., a),$$

for all $n \in \mathbb{N}$ and for any arbitrary $d_1, ..., d_n \in D_N$. So, the relations (1.2) and (1.3) shows that it is enough to consider the trivial D_N-valued moments $E(a^n)$ and cumulants $k_n(a, ..., a)$ of D_N-valued random variables ($n \in \mathbb{N}$), whenever we want to know about the free probabilistic information of those D_N-valued random variables.

Proposition 1.1. *Let (A, E) be the direct producted noncommutative probability space with amalgamation over the N-th diagonal algebra D_N, where $A = \oplus_{j=1}^{N} A_j$ and let a_1 and a_2 be D_N-valued random variables in (A, E). Then they are free over D_N in (A, E) if and only if all mixed **trivial** D_N-valued cumulants of them vanish.* $\square$

Remark 1.1. *If we have a direct producted $*$-probability (or C^*-probability or W^*-probability) space (A, E), then the above proposition does not hold, in general. But, we can get that the D_N-valued random variables a_1 and a_2 are free over D_N in (A, E) if and only if all mixed trivial D_N-valued cumulants of a_1, a_1^*, a_2 and a_2^* vanish.* $\square$

We will consider the D_N-valued moments of an arbitrary random variable in the direct producted noncommutative probability space.

Proposition 1.2. *Let $\left(A = \oplus_{j=1}^{N} A_j, \ E = \oplus_{j=1}^{N} \varphi_j\right)$ be the direct producted noncommutative probability space over the N-th diagonal algebra D_N and let $x = \oplus_{j=1}^{N} a_j$ be the D_N-valued random variable in (A, E). Then the trivial n-th moment of x is*

$$(1.4) \qquad E(x^n) = \oplus_{j=1}^{N} \left(\varphi_j(a_j^n)\right),$$

for all $n \in \mathbb{N}$. $\square$

The above proposition is easily proved by the fact that

$$x^n = \left(\oplus_{j=1}^{N} a_j\right)^n = \oplus_{j=1}^{N} a_j^n, \text{ for all } n \in \mathbb{N}.$$

It shows that if we know the n-th moments of $a_j \in (A_j, \varphi_j)$, for $j = 1, ..., N$, then we can compute the D_N-valued moment of the D_N-valued random variable $\oplus_{j=1}^{N} a_j$ in (A, E). In other words, the free probabilistic information of $A_1, ..., A_N$ affects the amalgamated free probabilistic information of $\oplus_{j=1}^{N} A_j$. Now, let's compute the trivial n-th cumulant of an arbitrary D_N-valued random variable;

Proposition 1.3. *Let* $\left(A = \oplus_{j=1}^{N} A_j, E = \oplus_{j=1}^{N} \varphi_j\right)$ *be the direct produced noncommutative probability space over the N-th diagonal algebra D_N and let $x = \oplus_{j=1}^{N} a_j$ be the D_N-valued random variable in (A, E). Then the trivial n-th cumulant of x is*

$$k_n \left(\underbrace{x, ..., x}_{n\text{-}times} \right) = \oplus_{j=1}^{N} \left(k_n^{(j)} \left(a_j, ..., a_j \right) \right),$$

for all $n \in \mathbb{N}$, where $k_n^{(i)}(...)$ is the n-th cumulant functional with respect to the noncommutative probability space (A_i, φ_i), for all $i = 1, ..., N$.

Proof. Fix $n \in \mathbb{N}$. Then

$$k_n (x, ..., x) = \sum_{\pi \in NC(n)} E_\pi (x, ..., x) \mu(\pi, 1_n)$$

(1.5)

$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} E_V (x, ..., x) \right) \mu(\pi, 1_n)$$

by (1,1), where $E_V(x, ..., x) = E(x^{|V|})$, where $|V|$ is the length of the block (See [1] and [11])

$$= \sum_{\pi \in NC(n)} \left(\prod_{V \in \pi} \left(\oplus_{j=1}^{N} (\varphi_j(a_j^{|V|})) \right) \right) \mu(\pi, 1_n)$$

by (1.4)

$$= \sum_{\pi \in NC(n)} \left(\oplus_{j=1}^{N} \left(\prod_{V \in \pi} \varphi_j(a_j^{|V|}) \right) \right) \mu(\pi, 1_n)$$

since $\left(\oplus_{j=1}^N \alpha_j\right) \cdot \left(\oplus_{j=1}^N \alpha'_j\right) = \oplus_{j=1}^N \alpha_j \alpha'_j$ in D_N

$$= \sum_{\pi \in NC(n)} \left(\oplus_{j=1}^N \left((\underset{V \in \pi}{\Pi} \varphi_j(a_j^{|V|})) \mu(\pi, 1_n) \right) \right),$$

since the N-th diagonal algebra D_N is a vector space (i.e., if we let $\mu_\pi = \mu(\pi, 1_n)$ in $\mathbb{C}$, for the fixed $\pi \in NC(n)$, then $\left(\oplus_{j=1}^N \alpha_j\right) \cdot \mu_\pi = \oplus_{j=1}^N (\alpha_j \mu_\pi)$, for all $\oplus_{j=1}^N \alpha_j \in D_N$.)

$$= \oplus_{j=1}^N \left(\sum_{\pi \in NC(n)} \left(\underset{V \in \pi}{\Pi} \varphi_j(a_j^{|V|}) \right) \mu(\pi, 1_n) \right)$$

since $(\alpha_1, ..., \alpha_N) + (\alpha'_1, ..., \alpha'_N) = (\alpha_1 + \alpha'_1, ..., \alpha_N + \alpha'_N)$ in D_N.

(1.6)

$$= \oplus_{j=1}^N \left(k_n^{(j)}(a_j, ..., a_j) \right),$$

where $k_n^{(i)}(...)$ is the (scalar-valued) n-th cumulant functional with respect to the (scalar-valued) noncommutative probability space (A_i, φ_i), for all $i = 1, ..., N$. ∎

Therefore, the equality (1.6) shows that the n-th D_N-valued cumulant

$$k_n \left(\oplus_{j=1}^N a_j, ..., \oplus_{j=1}^N a_j \right)$$

of the D_N-valued random variable $\oplus_{j=1}^N a_j$ in the direct producted noncommutative probability space (A, E), is nothing but the N-tuple of n-th (scalar-valued) cumulants of $a_1, ..., a_N$,

$$\oplus_{j=1}^N \left(k_n^{(j)}(a_j, ..., a_j) \right)$$

in D_N. By (1.6), we can get the following D_N-freeness characterization on the direct producted noncommutative probability space

$$\left(A = \oplus_{j=1}^N A_j, \ E = \oplus_{j=1}^N \varphi_j \right).$$

Theorem 1.4. Let $\left(A = \oplus_{j=1}^N A_j, E = \oplus_{j=1}^N \varphi_j \right)$ be the given direct producted non-commutative probability space over the N-th diagonal algebra D_N and let $x_1 = \oplus_{j=1}^N a_j$ and $x_2 = \oplus_{j=1}^N b_j$ be the D_N-valued random variables in (A, E). Then x_1 and x_2 are free over D_N in (A, E) if and only if a_j and b_j are free in (A_j, φ_j), for all $j = 1, ..., N$.

Proof. ($\Leftarrow$) Assume that random variables a_j and b_j are free in (A_j, φ_j), for all j = 1, ..., N. Then, by the freeness, all mixed n-th cumulants of a_j and b_j vanish, for all $j = 1$, ..., N and for all $n \in \mathbb{N} \setminus \{1\}$. By the previous theorem, it is sufficient to show that the N-tuples $x_1 = \oplus_{j=1}^{N} a_j$ and $x_2 = \oplus_{j=1}^{N} b_j$ in (A, E) have vanishing mixed **trivial** D_N-valued cumulants. Fix $n \in \mathbb{N} \setminus \{1\}$ and let $(x_{i_1}, ..., x_{i_n})$ are mixed n-tuple of x_1 and x_2, where $(i_1, ..., i_n) \in \{1, 2\}^n$. Then

$$k_n(x_{i_1}, ..., x_{i_n}) = k_n \left(\oplus_{j=1}^{N} a_j^{i_1}, ..., \oplus_{j=1}^{N} a_j^{i_n} \right)$$

$$= \oplus_{j=1}^{N} \left(k_n^{(j)}(a_j^{i_1}, ..., a_j^{i_n}) \right)$$

by the previous proposition

$$= 0_{D_N},$$

by the hypothesis.

($\Rightarrow$) Let's assume that the D_N-valued random variables $x_1 = \oplus_{j=1}^{N} a_j$ and $x_2 = \oplus_{j=1}^{N} b_j$ are free over D_N in (A, E) and assume also that there exists j in $\{1, ..., N\}$ such that a_j and b_j are not free in (A_j, φ_j). Since a_j and b_j are not free in (A_j, φ_j), there exists $n \in \mathbb{N}$ and the mixed n-tuple $(i_1, ..., i_n) \in \{1, 2\}^n$ such that $k_n^{(j)}(a_j^{i_1}, ..., a_j^{i_n}) = \beta \neq 0$. Now, fix the number n and the n-tuple $(i_1, ..., i_n)$. Consider the following mixed trivial D_N-valued cumulants of x_1 and x_2;

$$k_n(x_{i_1}, ..., x_{i_n}) = k_n \left(\oplus_{j=1}^{N} a_j^{i_1}, ..., \oplus_{j=1}^{N} a_j^{i_n} \right)$$

$$= \oplus_{j=1}^{N} \left(k_n^{(j)}(a_j^{i_1}, ..., a_j^{i_n}) \right)$$

$$= 0 \oplus ... \oplus 0 \oplus \underset{j\text{-th}}{\boxed{\beta}} \oplus 0 \oplus ... \oplus 0 \neq 0_{D_N}.$$

Therefore, there exists the nonvanishing D_N-valued cumulant of x_1 and x_2. This contradict our assumption that D_N-valued random variables x_1 and x_2 are free over D_N in (A, E). ∎

Remark 1.2. *Now, assume that above (A, E) is direct producted $*$-probability (or C^*-probability or W^*-probability) space of (A_1, φ_1), ..., (A_N, φ_N), where (A_j, φ_j)'s are $*$-probability spaces, for all $j = 1$, ..., N. Then the same results holds true. But the proof should be different, by the previous remark. Assume that $x_1 = \oplus_{j=1}^{N} a_j$ and $x_2 = \oplus_{j=1}^{N} b_j$ in (A, E) are free over D_N in (A, E). Then $*$-$Alg(\{x_1\}, D_N)$ and $*$-$Alg(\{x_2\}, D_N)$ are free over D_N, where $*$-$Alg(S_1, S_2)$ means the $*$-algebra generated by sets S_1 and S_2. Notice that*

$$*\text{-}Alg\left(\{x_1\}, D_N\right) = D_N \oplus \left(\oplus_{j=1}^{N} (*\text{-}Alg(\{a_j\})) \right)$$

and

$$*\text{-}Alg(\{x_2\}, D_N) = D_N \oplus \left(\oplus_{j=1}^N (*\text{-}Alg(\{b_j\}))\right),$$

as $*$-subalgebras in A. Assume now that a_{j_0} and b_{j_0} are not free in (A_{j_0}, φ_{j_0}), for some $j_0 \in \{1, ..., N\}$. Then the direct summand

$$D_1 = D_N \oplus \left(0 \oplus ... \oplus 0 \oplus \underset{j_0\text{-th}}{*\text{-}Alg(\{a_{j_0}\})} \oplus 0 \oplus ... \oplus 0\right)$$

and

$$D_2 = D_N \oplus \left(0 \oplus ... \oplus 0 \oplus \underset{j_0\text{-th}}{*\text{-}Alg(\{b_{j_0}\})} \oplus 0 \oplus ... \oplus 0\right)$$

are not free over D_N in (A, E). Since $*\text{-}Alg(\{x_i\}, D_N)$ contain D_i, for $i = 1$, 2, and since D_1 and D_2 are not free over D_N, $*\text{-}Alg(\{x_1\}, D_N)$ and $*\text{-}Alg(\{x_2\}, D_N)$ are not free over D_N. This contradict our assumption. The converse is also similarly proved. Also, we provide the following proof of the converse;

Since a_j and b_j are free in (A_j, φ_j), for all $j = 1, ..., N$, all mixed cumulants of a_j, a_j^*, b_j and b_j^* vanish, for all $j = 1, ..., N$. So, it suffices to show that all mixed trivial D_N-valued cumulants of x_1, x_1^*, x_2 and x_2^* vanish. Notice that

$$(1.7) \quad k_n\left(x_{i_1}^{e_1}, ..., x_{i_n}^{e_n}\right) = \oplus_{j=1}^N \left(k_n^{(j)}(a_{i_1:j}^{e_{i_1}}, ..., a_{i_n:j}^{e_{i_n}})\right),$$

where $x_{i_k}^{e_{i_k}} = \oplus_{j=1}^N \left(a_{i_k:j}^{e_{i_k}}\right)$ in (A, E), for all $k = 1, ..., n$, and where $e_{i_1}, ..., e_{i_n} \in \{1, *\}$ and $(i_1, ..., i_n) \in \{1, 2\}^n$, for all $n \in \mathbb{N}$. Therefore, for the mixed n-tuple $(x_{i_1}^{e_1}, ..., x_{i_n}^{e_{i_n}})$ of $\{x_1, x_1^*, x_2, x_2^*\}$, the n-tuples $(a_{i_1:k}^{e_{i_1}}, ..., a_{i_n:k}^{e_{i_n}})$ are also mixed n-tuple of $\{a_k, a_k^*, b_k, b_k^*\}$, for all $k = 1, ..., N$. Therefore, for such mixed n-tuple,

$$(1.8) \qquad k_n^{(k)}\left(a_{i_1:k}^{e_{i_1}}, ..., a_{i_n:k}^{e_{i_n}}\right) = 0, \text{ for all } k = 1, ..., N.$$

By (1.8), the n-th mixed trivial D_N-valued cumulants of x_1, x_1^*, x_2 and x_2^* in (1.7) vanish, and hence the D_N-valued random variables x_1 and x_2 are free over D_N in the direct producted $*$-probability space (A, E).

The above theorem and remark shows that the D_N-freeness of $\oplus_{j=1}^N a_j$ and $\oplus_{j=1}^N b_j$ in the direct producted noncommutative (or $*$- or C^*- or W^*-) probability space (A, E) is characterized by the (scalar-valued) freeness of a_j and b_j in (A_j, φ_j), for all $j = 1, ..., N$.

Corollary 1.5. *Let $e_i = 0 \oplus ... \oplus 0 \oplus a_i \oplus 0 \oplus ... \oplus 0$ and $e_j = 0 \oplus ... \oplus 0 \oplus a_j \oplus 0 \oplus ... \oplus 0$ in (A, E), where $a_k \in (A_k, \varphi_k)$, for $k = i, j$. If $i \neq j$, then e_i and e_j are free over D_N in (A, E).* $\square$

Define subalgebras $A_1', ..., A_N'$ of the direct product $A = \oplus_{j=1}^N A_j$ by

$$A'_j = 0 \oplus \dots \oplus 0 \oplus A_j \oplus 0 \oplus \dots \oplus 0,$$

for all $j = 1, ..., N$. Then A'_j is the embedding of A_j in A. By the previous corollary, we can easily get the following proposition;

Corollary 1.6. *The unital algebras A_1, ..., A_N are free over D_N in the direct produced noncommutative probability space (A, E).* $\square$

In the above corollary, we can replace the condition [algebras A_1, ..., A_N] to [∗-algebras A_1, ..., A_N]. By definition, the unital algebra A_j is always free from $\mathbb{C}$ (See [1], [4], [10] and [11]).

2. Graph Free Probability Spaces

In this chapter, we will consider our main objects of this paper. Throughout this chapter, let G be a finite simplicial graph with its finite vertex set $V(G)$ and the edge set $E(G)$. Let $e \in E(G)$ be an edge connecting the vertices v_1 and v_2. Then denote e by $[v_1, v_2]$. Assume that $[v_1, v_2] = [v_2, v_1]$, if there exists an edge e connecting v_1 and v_2. Since the graph G is simplicial, if $[v_1, v_2]$ is in $E(G)$, then this is the unique edge connecting the vertices v_1 and v_2. By $FP(G)$, we will denote the set of all admissible finite paths. Then this set $FP(G)$ is partitioned by

$$FP(G) = \cup_{n=1}^{\infty} FP_n(G),$$

with

$$FP_n(G) = \{w \in FP(G) : |w| = n\},$$

where $|w|$ is the length of the finite path w in $FP(G)$. i.e., if $w = e_1\, e_2\, \dots\, e_k$, where $e_1 = [v_1, v_2]$, $e_2 = [v_2, v_3]$, ..., $e_k = [v_k, v_{k+1}]$ are admissible edges making the finite path w, then the path w is denoted by $[v_1, v_2, ..., v_{k+1}]$, and $|w|$ is defined to be k. It means that the finite path w connects vertices v_1 and v_{k+1} via v_2, ..., v_{k-1}, and the length $|w|$ of w is $k = |\{v_1, ..., v_{k+1}\}| - 1$. And then $w \in FP_k(G)$ in $FP(G)$. Clearly, the edge set $E(G)$ is $FP_1(G)$. The following is automatically assumed;

$$[v_1, v_2, ..., v_{n-1}, v_n] = [v_n, v_{n-1}, ..., v_2, v_1],$$

for all $n \in \mathbb{N} \setminus \{1\}$, where $v_1, ..., v_n \in V(G)$. Define the free semigroupoid $\mathbb{F}^+(G)$ by

$$\mathbb{F}^+(G) = V(G) \cup FP(G).$$

2.1. **Graph Free Product of Probability-Space-Vertices.**

Throughout this section, let G be a finite simplicial graph, having its vertex-set $V(G) = \{1, ..., N\}$, with the probability-space vertices $(A_1, \varphi_1), ..., (A_N, \varphi_N)$. i.e., there exists a set of noncommutative probability spaces, indexed by the given vertex-set $V(G)$,

$$\{(A_1, \varphi_1), ..., (A_N, \varphi_N)\}.$$

Such graphs with probability-space vertices are said to be graphs of probability-space vertices.

Definition 2.1. *Let G be a graph with probability-space vertices $\{(A_v, \varphi_v) : v \in V(G)\}$. Define the graph free product A^G of $\{A_v\}_{v \in V(G)}$, by the algebra*

$$A^G = \mathbb{C} \oplus \left(\underset{w \in \mathbb{F}^+(G)}{\oplus} A_w \right),$$

where $A_w = A_w$, for all $w \in V(G)$, and

$$A_w = A_{v_1} * A_{v_2} * ... * A_{v_k},$$

for all $w = [v_1, v_2, ..., v_k] \in FP(G)$, $k \in \mathbb{N}$. If A_w is a summand of A^G, we write $A_w <_G A^G$. Sometimes, we will denote A^G by $\underset{v \in V(G)}{*^G} A_v$. The symbol "$*^G$" is called the graph free product. Since we have the free product $\varphi_w = *_{n=1}^{k} \varphi_{v_n}$ of linear functionals $\varphi_{v_1}, ..., \varphi_{v_n}$, for each finite path $w = [v_1, ..., v_k]$, we have the corresponding noncommutative probability space (A_w, φ_w). (Here, the symbol "$*$" means the usual free product.) Now, define the G-diagonal subalgebra D^G of A^G by

$$D^G = \mathbb{C} \oplus \left(\underset{w \in \mathbb{F}^+(G)}{\oplus} \mathbb{C}_w \right),$$

where $\mathbb{C}_v = \mathbb{C} = \mathbb{C}_w$, for all $v \in V(G)$ and $w \in FP(G)$. Then, like in Chapter 1, we can define the conditional expectation E^G from A^G into D^G by

$$E^G = \left(\underset{w \in \mathbb{F}^+(G)}{\oplus} \varphi_w \right).$$

*Then the algebraic pair $\left(A^G, E^G \right)$ is a noncommutative probability space with amalgamation over D^G, and it is called the **graph free probability space** of $\{(A_v, \varphi_v) : v \in V(G)\}$. All elements in the graph free probability space (A^G, E^G) are called graph random variables or G-random variables.*

By definition, we can easily check that the subalgebra D^G commutes with A^G. i.e.,

(2.1) $da = ad$, for all $d \in D^G$ and $a \in A^G$.

By definition and (2.1), we have that;

Proposition 2.1. *Let (A^G, E^G) be a graph free probability space over its subalgebra D^G. Then it is a direct producted noncommutative probability space of $\{(A_w, \varphi_w) : w \in \mathbb{F}^+(G)\}$.* $\square$

Definition 2.2. *(**Graph-Freeness**) Let G be a graph with probability-space vertices and let (A^G, E^G) be the corresponding graph free probability space over its G-diagonal subalgebra D^G. The subalgebras A_1 and A_2 of A^G are said to be graph-free or G-free if all mixed D^G-valued cumulants of A_1 and A_2 vanishes. The subsets X_1 and X_2 of A^G are G-free if the subalgebras Alg (X_1, D^G) and Alg (X_2, D^G) are G-free.*

By the previous definition, we have that;

Theorem 2.2. *Let x and y be G-random variables in (A^G, E^G). The G-random variables x and y are G-free if and only if either (i) or (ii) holds;*

(i) there exists $(A_w, \varphi_w) <_G (A^G, E^G)$ such that x and y are free in (A_w, φ_w).
(ii) they are free over D^G in (A^G, E^G).

Proof. Let x and y be G-random variables in the graph free probability space (A^G, E^G). Then, by the definition of A_G,

$$x = \underset{w \in \mathbb{F}^+(G)}{\oplus} x_w \quad \text{and} \quad y = \underset{w' \in \mathbb{F}^+(G)}{\oplus} y_{w'}.$$

($\Rightarrow$) Assume that the G-random variables x and y are G-free. i.e., they have the vanishing mixed D^G-valued cumulants. By Chapter 1, all summands x_w and $y_{w'}$ of x and y are free in (A_w, φ_w). So, if there exists $w_0 \in \mathbb{F}^+(G)$ such that $x = x_{w_0}$ and $y = y_{w_0}$, then, as scalar-valued random variables, x and y are free in $(A_{w_0}, \varphi_{w_0}) <_G (A^G, E^G)$. Otherwise, again by Chapter 1, they are free over D^G in (A^G, E^G), as D^G-valued random variables in the direct producted noncommutative probability space A^G of A_w's, for $w \in \mathbb{F}^+(G)$.

($\Leftarrow$) Suppose the G-random variables x and y are free in (A_w, φ_w), where $A_w <_G A^G$. First, this means that the operators x and y are contained in the summand A_w of A^G. Also, since they are free in (A_w, φ_w), they are free over D^G in (A^G, E^G), as D^G-valued random variables $x = x_w$ and $y = y_w$. Thus they are G-free. Otherwise, if x and y are free over D^G in (A^G, E^G), then, by the very definition of G-freeness, they are G-free in (A^G, D^G). $\blacksquare$

2.2. **Graph-Freeness.**

In this section, we will consider the graph-freeness more in detail. Throughout this section, let G be a graph with probability-space vertices

$$\{(A_v, \varphi_v) : v \in V(G)\}.$$

Also, let (A^G, E^G) be the corresponding graph free probability space over its G-diagonal subalgebra D^G.

Lemma 2.3. *Let A_{v_1} and A_{v_2} be the direct summands of A^G, where $v_1 \neq v_2$ in $V(G)$. Then they are G-free in (A^G, E^G).*

Proof. In Chapter 1, we showed that the direct summands of a direct producted noncommutative probability space are free from each other over the diagonal subalgebra. So, if $v_1 \neq v_2$ in $V(G)$, then A_{v_1} and A_{v_2} are free over D^G in (A^G, E^G), as direct summands of A^G. Equivalently, they are G-free in (A^G, E^G). ∎

We will consider more general case. To do that, we need the following concept;

Definition 2.3. *Let $\mathbb{F}^+(G)$ be the free semigroupoid of the given graph G and let w_1 and w_2 be elements in $\mathbb{F}^+(G)$. We say that $w_1 = [v_1, ..., v_k]$ and $w_2 = [v'_1, ..., v'_l]$ are disjoint if $\{v_1, ..., v_k\} \cap \{v'_1, ..., v'_l\} = \emptyset$, for $k, l \in \mathbb{N}$. (Notice that if $k = 1$ and $l = 1$, then $w_1 = [v_1]$ and $w_2 = [v'_1]$ are vertices in $V(G)$.)*

Theorem 2.4. *Let $B_{w_1} \simeq A_{w_1}$ and $B_{w_2} \simeq A_{w_2}$ be subalgebras of A^G generated by w_1 and w_2, respectively. (Notice that A_{w_1} and A_{w_2} are direct sums of A^G, but B_{w_1} and B_{w_2} are not necessarily direct sums. They are just subalgebras of A^G.) The subalgebras B_{w_1} and B_{w_2} are G-free if and only if either (i) $B_{w_1} = A_{w_1}$ and $B_{w_2} = A_{w_2}$ or (ii) w_1 and w_2 are disjoint.*

Proof. ($\Leftarrow$) If $B_{w_1} = A_{w_1}$ and $B_{w_2} = A_{w_2}$, then they are G-free, by definition. If $w_1 = [v_1, ..., v_k]$ and $w_2 = [v'_1, ..., v'_l]$ are disjoint $(k, l \in \mathbb{N})$, then the direct summands A_{w_1} and A_{w_2} of the graph free product A^G have no common direct summands. This means that, for any pair $(v_i, v'_j) \in \{v_1, ..., v_k\} \times \{v'_1, ..., v'_l\}$, the direct summands A_{v_i} and $A_{v'_j}$ are G-free, by the previous lemma. Therefore,

$$B_{w_1} = A_{v_1} * ... * A_{v_k} \quad \text{and} \quad B_{w_2} = A_{v'_1} * ... * A_{v'_l}$$

are free over D^G in (A^G, E^G), and hence they are G-free.

$(\Rightarrow)$ Suppose B_{w_1} and B_{w_2} are G-free in (A^G, E^G) and assume that $w_1 = [v_1, ..., v_k]$ and $w_2 = [v'_1, ..., v'_l]$ are not disjoint in $\mathbb{F}^+(G)$, where $k, l \in \mathbb{N}$. Then, by definition, there exists the nonempty intersection of $\{v_1, ..., v_k\}$ and $\{v'_1, ..., v'_l\}$. Take a vertex v_0 in the intersection. Then the algebra A_{v_0} of A^G is contained in both A_{w_1} and A_{w_2}. i.e.,

$$B_{w_1} = A_{v_0} * \left(\underset{v \in \{v_1, ..., v_k\} \setminus \{v_0\}}{*} A_v \right)$$

and

$$B_{w_2} = A_{v_0} * \left(\underset{v' \in \{v'_1, ..., v'_l\} \setminus \{v_0\}}{*} A_{v'} \right).$$

(Recall that $A * B = B * A$, for algebras A and B.) We can take $x_0 \in A_{v_0} \setminus \mathbb{C}$ such that $\varphi_{v_0}(x_0^k) \neq 0$, for some $k \in \mathbb{N}$. Define $x = x_0 \in B_{w_1}$ and $y = x_0 \in B_{w_2}$. Then the random variables x and y have the nonvanising mixed cumulants, with respect to the linear functional φ_{v_0}. Therefore, by regarding them as G-random variables, they have the nonvanishing mixed D^G-valued cumulants, with respect to the conditional expectation E^G. This shows that B_{w_1} and B_{w_2} are not free over D^G in the direct producted noncommutative probability space (A^G, E^G). Equivalently, they are not G-free. This contradict our assumption that B_{w_1} and B_{w_2} are G-free. $\blacksquare$

By the previous theorem, we can get the following corollaries;

Corollary 2.5. *Let W_1 and W_2 are subsets in $\mathbb{F}^+(G)$ and assume that W_1 and W_2 are disjoint, in the sense that all pairs $(w_1, w_2) \in W_1 \times W_2$ are disjoint. Then the subalgebras $\underset{w_1 \in W_1}{\oplus} A_{w_1}$ and $\underset{w_2 \in W_2}{\oplus} A_{w_2}$ are G-free in (A^G, E^G). And the converse also holds true.* $\square$

Corollary 2.6. *Let $w = [v_1, ..., v_j, ..., v_k]$ be a finite path, for $1 \leq j < k$ in $\mathbb{N}$. Then $A_{[v_1, ..., v_j]}$ and $A_{[v_{j+1}, ..., v_k]}$ are G-free in (A^G, E^G).*

Proof. Notice that since $[v_1, ..., v_k]$ is a finite path, $[v_1, ..., v_j]$ and $[v_{j+1}, ..., v_k]$ are also in $\mathbb{F}^+(G)$. Since $[v_1, ..., v_j]$ and $[v_{j+1}, ..., v_k]$ are disjoint, as direct summands of A^G, the algebras $A_{[v_1, ..., v_j]}$ and $A_{[v_{j+1}, ..., v_k]}$ are G-free. $\blacksquare$

2.3. Graph R-transform Calculus.

In this section, we will define graph moment series and graph R-transforms of graph random variables. Like before, let G be a graph with probability-space

vertices and let (A^G, E^G) be the corresponding graph free probability space over the G-diagonal subalgebra D^G, where $A^G = \underset{w \in \mathbb{F}^+(G)}{*^G} A_w$ and $E^G = \underset{w \in \mathbb{F}^+(G)}{\oplus} \varphi_w$.

Definition 2.4. *Let (A^G, E^G) be a graph free probability space over the G-diagonal sublagebra D^G and let $x \in (A^G, E^G)$ be a G-random variable. The graph moments (or G-moments) of x and the graph cumulants (or G-cumulants) of x are defined by*

$$E^G(x^n) \quad and \quad k_n^{(E^G)} \left(\underbrace{x, ...,x}_{n\text{-}times} \right),$$

for all $n \in \mathbb{N}$.

Let x be a G-random variable in (A^G, E^G). Then $x = \underset{w \in \mathbb{F}^+(G)}{\oplus} x_w$. So, by Chapter 1, we have that

$$(2.2) \qquad E^G(x^n) = \underset{w \in \mathbb{F}^+(G)}{\oplus} (\varphi_w(x_w^n)),$$

and

$$(2.3) \qquad k_n^{(E^G)}(x, ..., x) = \underset{w \in \mathbb{F}^+(G)}{\oplus} \left(k_n^{(\varphi_w)}(x_w, ..., x_w) \right),$$

for all $x = \underset{w \in \mathbb{F}^+(G)}{\oplus} x_w \in A^G$ and for all $n \in \mathbb{N}$. Notice that, by (2.1), we have that

$$k_n^{(E^G)}(x, ..., x) = \sum_{\pi \in NC(n)} \left(\underset{V \in \pi}{\Pi} E^G(x^{|V|}) \right) \mu(\pi, 1_n),$$

like the scalar-valued cumulants. Recall that, when we consider the operator-valued Möbius inversion, we have to think about the insertion property. However, in our case, by (2.1), we need not think about the insertion property. Also, the trivial D^G-valued cumulants (or moments) of x contains the full free distributional data of x.

Definition 2.5. *Let $x \in (A^G, E^G)$ be a G-random variable. Define the graph moment series (or G-moment series) of x by*

$$M_x^G(z) = \sum_{n=1}^{\infty} \left(E^G(x^n) \right) z^n$$

in the ring $D^G[[z]]$ of formal series, where z is an indeterminent.. Also, define the graph R-transform (or G-R-transform) of x by

$$R_x^G(z) = \sum_{n=1}^{\infty} \left(k_n^{(E^G)}(x, ..., x) \right) z^n$$

in $D^G[[z]]$. More generally, if x_1, ..., x_s are G-random variables, then the G-moment series of x_1, ..., x_s and G-R-transform of them are defined by

$$M_{x_1,...,x_s}^G(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{(i_1,...,i_n) \in \{1,...,s\}^n} E^G(x_{i_1}...x_{i_n}) \, z_{i_1}...z_{i_n}$$

and

$$R_{x_1,...,x_s}^G(z_1, ..., z_s) = \sum_{n=1}^{\infty} \sum_{(i_1,...,i_n) \in \{1,...,s\}^n} k_n^{(E_G)}(x_{i_1}, ..., x_{i_n}) \, z_{i_1}...z_{i_n},$$

in $D^G[[z_1, ..., z_s]]$, where z_1, ..., z_s are noncommutative indeterminants.

By (2.1), the graph moment series and graph R-transforms of graph random variables are well-defined.

Proposition 2.7. *Let x and y be G-random variables and assume that they are G-free. Then*

(1) $R_{x+y}^G(z) = \left(R_x^G + R_y^G \right)(z)$

(2) $R_{x,y}^G(z_1, z_2) = R_x^G(z_1) + R_y^G(z_2).$ $\square$

More generally, we have that;

Theorem 2.8. *Let $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$ be sets of G-random variables ($s \in \mathbb{N}$) in the graph free probability space (A^G, E^G). If these two sets are G-free, then*

(1) $R_{x_1+y_1,...,x_s+y_s}^G(z_1, ..., z_s) = \left(R_{x_1,...,x_s}^G + R_{y_1,...,y_s}^G \right)(z_1, ..., z_s)$

(2) $R_{x_1,...,x_s,y_1,...,y_s}^G(z_1, .., z_{2s}) = R_{x_1,...,x_s}^G(z_1, ..., z_s) + R_{y_1,...,y_s}^G(z_{s+1}, ..., z_{2s}).$ $\square$

Let x and y be G-random variables in the graph free probability space (A^G, E^G) and assume that they are G-free. Then they are free over D^G in (A^G, E^G), by regarding (A^G, E^G) as a direct producted noncommutative probability space. So, we have that

$$k_n^{(E^G)}(xy, ..., xy)$$

(2.4)
$$= \sum_{\pi \in NC(n)} \left(k_\pi^{(E^G)}(x, ..., x) \right) \left(k_{Kr(\pi)}^{(E^G)}(y, ..., y) \right),$$

for $n \in \mathbb{N}$, where $Kr : NC(n) \to NC(n)$ is the Kreweras complementation map on $NC(n)$. In fact,

(2.5)

$$k_n^{(E^G)}(xy, ..., xy) = \sum_{\pi \in NC(n)} k_{\pi \cup_{alt} Kr(\pi)}^{(E_G)}(x, y, x, y, ..., x, y),$$

by the D^G-freeness of x and y, where $\pi \cup_{alt} \theta$ means the alternating sum of partitions π and θ in $NC(n)$, $n \in \mathbb{N}$. i.e., if

$$\pi = \{(1, 4, 5), (2, 3), (6, 8), (7)\} \in NC(8),$$

then

$$Kr(\pi) = \{(1, 3), (2), (4), (5, 8), (6, 7)\} \in NC(8)$$

and

$$\pi \cup_{alt} Kr(\pi) = \quad \{(1, 7, 9), (2, 6), (3, 5), (4),$$
$$(10, 16), (11, 15), (12, 14), (13)\} \in NC(16).$$

However, by (2.1), the formula (2.5) is exactly same as (2.4). So, the G-R-transform $R_{xy}^G(z)$ of xy satisfies that

(2.6)

$$R_{xy}^G(z) = \sum_{n=1}^{\infty} \left(\sum_{\pi \in NC(n)} \left(k_{\pi}^{(E^G)}(x, ..., x) \right) \left(k_{Kr(\pi)}^{(E^G)}(y, ..., y) \right) \right) z^n.$$

More generally, if two sets $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$ are G-free in (A^G, E^G), then

$$k_n^{(E^G)}(x_{i_1} y_{i_1}, x_{i_2} y_{i_2}, ..., x_{i_n} y_{i_n})$$

(2.7) $$= \sum_{\pi \in NC(n)} \left(k_{\pi}^{(E^G)}(x_{i_1}, ..., x_{i_n}) \right) \left(k_{Kr(\pi)}^{(E^G)}(y_{i_1}, ..., y_{i_n}) \right),$$

where $(i_1, ..., i_n) \in \{1, ..., s\}^n$, for $n \in \mathbb{N}$. So,

(2.8) $$R_{x_1 y_1, ..., x_s y_s}^G(z_1, ..., z_s) =$$

$$\sum_{n=1}^{\infty} \sum_{(i_1, ..., i_n) \in \{1, ..., s\}^n} \left(\sum_{\pi \in NC(n)} \left(k_{\pi}^{(E^G)}(x_{i_1}, ..., x_{i_n}) \right) \left(k_{Kr(\pi)}^{(E^G)}(y_{i_1}, ..., y_{i_n}) \right) \right) z_{i_1} ... z_{i_n}$$

Notation In the rest of this paper, the right-hand side of (2.6) is denoted by

$$\left(R_x^G \boxtimes_G R_y^G \right)(z).$$

And similarly, the right-hand side of (2.8) is denoted by

$$\left(R^G_{x_1,...,x_s} \boxtimes_G R^G_{y_1,...,y_s}\right)(z_1,...,z_s).$$

The symbol "$\boxtimes_G$" is motivated by the boxed convolution $\boxtimes_s$ of Nica. Recall that the Nica's boxed convolution $\boxtimes_s$ makes a certain subset Θ_s of $\mathbb{C}[[z_1,...,z_s]]$ be semigroup (See [1]). But our boxed-convolution-like notation $\boxtimes_G$ does not have such meaning. This is just a notation representing the right-hand sides of (2.6) and (2.8).
$\square$

By the previous discussion, we have that;

Proposition 2.9. *Let $\{x_1,...,x_s\}$ and $\{y_1,...,y_s\}$ be G-free in the G-free probability space (A^G, E^G). Then*

$$R^G_{x_1 y_1,...,x_s y_s}(z_1,...,z_s) = \left(R^G_{x_1,...,x_s} \boxtimes_G R^G_{y_1,...,y_s}\right)(z_1,...,z_s),$$

in $D^G[[z_1,...,z_s]]$. In particular, for each $j \in \{1, ..., s\}$,

$$R^G_{x_j y_j}(z) = \left(R^G_x \boxtimes_G R^G_y\right)(z)$$

in $D^G[[z]]$. $\square$

By the previous propositions, we have the following graph R-transform calculus; if $\{x_1, ..., x_s\}$ and $\{y_1, ..., y_s\}$ are G-free, then

(1) $R^G_{x_1+y_1,...,x_s+y_s}(z_1,...,z_s) = \left(R^G_{x_1,...,x_s} + R^G_{y_1,...,y_s}\right)(z_1,...,z_s)$
(2) $R^G_{x_1,...,x_s,y_1,...,y_s}(z_1,...,z_{2s}) = R^G_{x_1,...,x_s}(z_1,...,z_s) + R^G_{y_1,...,y_s}(z_{s+1},...,z_{2s})$
(3) $R^G_{x_1 y_1,...,x_s y_s}(z_1,...,z_s) = \left(R^G_{x_1,...,x_s} \boxtimes_G R^G_{y_1,...,y_s}\right)(z_1,...,z_s),$

in $D^G[[z_1,...,z_s]]$.

3. Graph-Free Random Variables

In this chapter, we will let each probability-space vertex be a W^*-probability space. Let G be a graph with W^*-probability-space vertices. i.e.,

$$\{(A_v, \varphi_v) : v \in V(G), \varphi_v \text{ is a state}\}$$

is a family of W^*-probability spaces. Define graph free product A^G of A_v's by

$$A^G = \mathbb{C}\overline{\oplus}\left(\overline{\underset{w \in \mathbb{F}^+(G)}{\oplus}} A_w\right),$$

where $\overline{\oplus}$ is the corresponding topological direct sum. Also, define the G-diagonal algebra D^G by

$$D^G = \mathbb{C}\overline{\oplus}\left(\overline{\underset{w \in \mathbb{F}^+(G)}{\oplus}} \mathbb{C}_w\right), \text{ with } \mathbb{C}_w = \mathbb{C}, \forall w.$$

Also, the conditional expectation is defined by $E^G = \overline{\underset{w \in \mathbb{F}^+(G)}{\oplus}} \varphi_w.$

> **Notation** For convenience, if there is no confusion, we will keep using the notation $\oplus$ instead of $\overline{\oplus}$. $\square$

Notice that the corresponding graph free probability space (A^G, E^G) is a W^*-probability space with amalgamation over the G-diagonal algebra D^G. In such amalgamated W^*-probability space (A^G, E^G), we will consider certain G-random variables. Notice that if x is a G-random variables in a graph free W^*-probability space (A^G, E^G), then

$$x = \underset{w \in \mathbb{F}^+(G)}{\oplus} x_w.$$

i.e., we can understand x is the (infinite) direct sum of x_w's, where $w \in \mathbb{F}^+(G)$.

3.1. Graph Semicircular Elements.

In this section, we will consider the graph-semicircularity of graph random variables. Let G be a finite simplicial graph with W^*-probability-space-vertices $\{(A_v, \varphi_v) : v \in V(G)\}$ and let (A^G, E^G) be the corresponding graph free W^*-probability space over its G-diagonal subalgebra D^G, with the G-conditional expectation $E^G = \underset{w \in \mathbb{F}^+(G)}{\oplus} \varphi_w.$

Definition 3.1. *Let $x \in (A^G, E^G)$ be a self-adjoint G-random variable. We say that this G-random variable x is graph-semicircular (or G-semicircular) if the only second G-cumulant of x is nonvanishing. i.e., x has the following G-cumulant relation;*

$$k_n^{(E^G)}\left(\underbrace{x, ..., x}_{n\text{-}times}\right) = \begin{cases} k_2^{(E^G)}(x,x) & \text{if } n = 2 \\ \\ 0_{D^G} & \text{otherwise.} \end{cases}$$

Equivalently, the G-random variable x is G-semicircular if x is D^G-valued semi-circular, as a D^G-valued random variable in the direct producted W^*-probability space (A^G, E^G). In the rest of this section, we will observe the conditions when the G-random variable x is G-semicircular.

Lemma 3.1. *Let (A_v, φ_v) be a W^*-probability-space-vertex of the graph G. If a $\in (A_v, \varphi_v)$ is (scalar-valued) semicircular, then, as a G-random variable a is G-semicircular.*

Proof. In general, if $x = \underset{w \in \mathbb{F}^+(G)}{\oplus} x_w$ is a G-random variable in (A^G, E^G), then

$$k_n^{(E^G)} \left(\underbrace{x, \dots\dots, x}_{n\text{-times}} \right) = \underset{w \in \mathbb{F}^+(G)}{\oplus} \left(k_n^{(\varphi_w)}(x_w, \dots, x_w) \right),$$

by Chapter 1 and (2.3). Thus, if $a \in (A_v, \varphi_v) <_G (A^G, E^G)$, then

$$(2.9) \qquad\qquad k_n^{(E^G)}(a, \dots, a) = k_n^{(\varphi_v)}(a, \dots, a).$$

Since a is semicircular in (A_v, φ_v), as a G-random variable, it is also G-semicircular, by (2.9). ■

By the previous lemma, we can get that;

Proposition 3.2. *Let $w = [v_1, \dots, v_k] \in FP(G)$ and let nonzero $a_{v_j} \in (A_{v_j}, \varphi_{v_j})$ be semicircular, for $j = 1, \dots, k$. Then the G-random variable $\sum_{j=1}^{k} a_{v_j}$ is G-semicircular in (A^G, E^G).*

Proof. By assumption, if the random variables a_{v_j}'s are nonzero, then they are semicircular in (A_{v_j}, φ_{v_j}), for some $j = 1, \dots, k$. Notice that $B_{v_1}, \dots, B_{v_k}$ are G-free in (A^G, E^G), where B_{v_j}'s are subalgebra isomorphic to the direct sums A_{v_j}, for all $j = 1, \dots, k$. Indeed, since $\{v_1\}, \dots, \{v_k\}$ are mutually disjoint, the subalgebras $B_{v_1}, \dots, B_{v_k}$ are G-free from each other. Therefore, as G-random variables $a_{v_1}, \dots, a_{v_k}$ are G-free from each other. So,

$$k_n^{(E^G)} \left(\sum_{j=1}^{N} a_{v_j}, \dots, \sum_{j=1}^{N} a_{v_j} \right)$$

$$= \sum_{j=1}^{N} k_n^{(E^G)} \left(a_{v_j}, \dots, a_{v_j} \right)$$

$$= \begin{cases} \sum_{j=1}^{N} k_2^{\varphi_{v_j}} \left(a_{v_j}, a_{v_j} \right) & \text{if } n = 2 \\ \\ 0_{D^G} & \text{otherwise} \end{cases}$$

by the previous lemma

$$
= \begin{cases}
k_2^{(E^G)} \left(\sum_{j=1}^N a_{v_j}, \ \sum_{j=1}^N a_{v_j} \right) & \text{if } n = 2 \\[2ex]
0_{D^G} & \text{otherwise,}
\end{cases}
$$

for all $n \in \mathbb{N}$. ∎

3.2. Graph Circular Elements.

In this section, we will consider the graph circularity on a graph free probability space (A^G, E^G).

Definition 3.2. *Let $a \in (A^G, E^G)$ be a G-random variable and assume that there exist self-adjoint G-random variables x_1 and x_2 in (A^G, E^G) such that $a = x_1 + ix_2$, and the G-random variables x_1 and x_2 are G-semicircular elements in (A^G, E^G). We say that the G-random variable a is G-circular if x_1 and x_2 are G-free in (A^G, E^G). We also say that the pair (x_1, x_2) is the G-semicircular pair of a.*

Suppose that G-circular element $a = x_1 + ix_2$ in (A^G, E^G) has G-semicircular elements x_1 and x_2 in (A_w, φ_w), for $w \in \mathbb{F}^+(G)$. Then the G-random variable a is also contained in (A_w, φ_w) and it is circular, in the sense of Voiculescu in this W^*-probability space (A_w, φ_w). Such G-random variable a is said to be a w-**circular element** in (A^G, E^G). By the G-freeness characterization, if w_1 and w_2 are disjoint in $\mathbb{F}^+(G)$, then $B_{w_1} \simeq A_{w_1}$ and $B_{w_2} \simeq A_{w_2}$ are G-free in (A^G, E^G). Let x_1 and x_2 be semicircular in (B_{w_1}, φ_{w_1}) and (B_{w_2}, φ_{w_2}), respectively. In this case, the G-random variable $a = x_1 + ix_2$ is again G-circular in (A^G, E^G). We say that such G-circular element a is (w_1, w_2)-**circular** in (A^G, E^G). Notice that if a is (w_1, w_2)-circular, then w_1 and w_2 are not admissible.

3.3. Graph R-diagonal Elements.

In this section, we define the graph R-diagonality on (A^G, E^G).

Definition 3.3. *Let $a \in (A^G, E^G)$ be a G-random variable. It is said to be G-R-diagonal if it has the following G-cumulant relation; the only nonvanishing G-cumulants of a are either*

$$k_{2n}^{(E^G)}\left(a, a^*, a, a^*, ..., a, a^*\right) \quad or \quad k_{2n}^{(E^G)}(a^*, a, a^*, a..., a^*, a),$$

for all $n \in \mathbb{N}$. We will call the above G-cumulants the alternating G-cumulants of a.

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